Topic 7 –
Second order linear homogeneous constant coefficient ODEs

In this topic we will learn how to find two linearly independent solutions to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

where a_2, a_1, a_0 are constants and $a_2 \neq 0$

Def: The <u>characteristic equation</u> of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$a_2 r^2 + a_1 r + a_0 = 0$$

There are three cases that can occur for the roots of the characteristic equation

<u>Ex of case 1:</u> distinct real roots

The characteristic equation of

$$2y'' - 5y' - 3y = 0$$

is

$$2r^2 - 5r - 3 = 0$$

which becomes

$$(2r+1)(r-3) = 0$$

which has two distinct real roots

$$r = -1/2, 3$$

Ex of two repeated real roots:

The characteristic eqvation of

$$y'' - 4y' + 4y = 0$$

is

$$r^2 - 4r + 4 = 0$$

which becomes

$$(r-2)(r-2) = 0$$

which has one repeated real root $r = 2$

Ex of two complex conjugate roots:

The characteristic polynomial of

$$y'' - 4y + 13y = 0$$

is

$$r^2 - 4m + 13 = 0$$

which has roots

$$r = \frac{-(-4) \pm \sqrt{16 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6\sqrt{-1}}{2} = 2 \pm 3i$$

Thus, we get two complex roots

$$r = 2+3i,\ r-3i$$

Let's analyze the cases starting with cases 1 & 2.
Suppose the characteristic equation

$$a_2 r^2 + a_1 r + a_0 = 0$$

of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

has a real root r.

Then,

$$a_2 r^2 + a_1 r + a_0 = 0$$

Consider the function $f(x) = e^{rx}$.

Then, $f'(x) = re^{rx}$, $f''(x) = r^2 e^{rx}$.

So, plugging f into the ODE gives

$$\begin{aligned} & a_2 f'' + a_1 f' + a_0 f \\ &= a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} \\ &= e^{rx}(a_2 r^2 + a_1 r + a_0) \\ &= e^{rx}(0) \\ &= 0 \end{aligned}$$

Thus, $f(x) = e^{rx}$ is a solution to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Case 1: Suppose r_1, r_2 are roots of the characteristic polynomial with $r_1 \neq r_2$. Then $f_1(x) = e^{r_1 x}$ and $f_2(x) = e^{r_2 x}$ both solve the ODE and the Wronskian is

$$W(e^{r_1 x}, e^{r_2 x}) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x}$$

$$= \underbrace{(r_2 - r_1)}_{r_2 - r_1 \neq 0} \underbrace{e^{(r_1+r_2)x}}_{e^{(r_1+r_2)x} > 0} \neq 0 \quad \text{for any } x$$

Thus, $f_1(x) = e^{r_1 x}$ and $f_2(x) = e^{r_2 x}$ are linearly independent and every solution to $a_2 y'' + a_1 y' + a_0 y = 0$ will be of the form

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Ex of case 1:

The ODE

$$2y'' - 5y' - 3y = 0$$

has characteristic polynomial

$$2m^2 - 5m - 3 = (2m+1)(m-3)$$

Since we have two distinct roots $-\frac{1}{2}, 3$ every solution to

$$2y'' - 5y' - 3y = 0$$

is of the form

$$y_h = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$$

Case 2: Suppose the characteristic polynomial of $a_2 y'' + a_1 y' + a_0 y = 0$ has only one real root r_1 but it's repeated.

We know one solution will be $f_1(x) = e^{r_1 x}$.

Let's show that another solution is $f_2(x) = xe^{r_1 x}$.

Since r_1 is a repeated root we get

r_1 is a repeated root

$$a_2 r^2 + a_1 r + a_0 = a_2 (r - r_1)^2$$
$$= a_2 r^2 - 2a_2 r_1 r + a_2 r_1^2$$

algebra

Thus, $a_1 = -2a_2 r_1$ and $a_0 = a_2 r_1^2$.

So the ODE becomes

$$a_2 y'' - 2a_2 r_1 y' + a_2 r_1^2 y = 0$$

Let's now plug in $f_2(x) = xe^{r_1 x}$.

We have

$$f_2(x) = xe^{r_1 x}$$
$$f_2'(x) = e^{r_1 x} + r_1 x e^{r_1 x}$$
$$f_2''(x) = r_1 e^{r_1 x} + r_1 e^{r_1 x} + r_1^2 x e^{r_1 x}$$

Plugging f_2 into the ODE gives

$$a_2 f_2'' - 2a_2 r_1 f_2' + a_2 r_1^2 f_2$$

$$= \cancel{a_2 r_1 e^{r_1 x}} + \cancel{a_2 r_1 e^{r_1 x}} + a_2 r_1^2 x e^{r_1 x}$$
$$- \cancel{2a_2 r_1 e^{r_1 x}} - 2a_2 r_1^2 x e^{r_1 x}$$
$$+ a_2 r_1^2 x e^{r_1 x}$$

$$= x e^{r_1 x}\left(a_2 r_1^2 - 2a_2 r_1 \cdot r_1 + a_2 r_1^2\right)$$

$$= x e^{r_1 x}\left(\underbrace{a_2 r_1^2 + a_1 r_1 + a_0}_{0}\right)$$

$$= 0$$

Thus, $f_2(x) = xe^{r_1 x}$ also solves the ODE.

The Wronskian of $f_1(x) = e^{r_1 x}$ and $f_2(x) = xe^{r_1 x}$ is

$$W(e^{r_1 x}, xe^{r_1 x}) = \begin{vmatrix} e^{r_1 x} & xe^{r_1 x} \\ r_1 e^{r_1 x} & e^{r_1 x} + r_1 x e^{r_1 x} \end{vmatrix}$$

$$= e^{2r_1 x} + r_1 x e^{2r_1 x} - r_1 x e^{2r_1 x}$$

$$= e^{2r_1 x} \neq 0 \quad \text{for any } x$$

Thus, $f_1(x) = e^{r_1 x}$ and $f_2(x) = x e^{r_1 x}$ are two linearly independent solutions to $a_2 y'' + a_1 y' + a_0 y = 0$ in this case and every solution must be of the form

$$y_h = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

Ex of case 2:

The characteristic equation of

$$y'' - 4y' + 4y = 0$$

is

$$r^2 - 4r + 4 = 0$$

which becomes

$$(r-2)(r-2) = 0$$

Thus, we have a repeated real root $r_1 = 2$.

So every solution of

$$y'' - 4y' + 4y = 0$$

is of the form

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

Case 3: Suppose the characteristic polynomial of $a_2y''+a_1y'+a_0y=0$ has two complex roots. We can divide by a_2 and we get the same equation $y''+\frac{a_1}{a_2}y'+\frac{a_0}{a_2}y=0$. For ease of derivation lets assume our equation has the form $y''+by'+cy=0$. And suppose we have two complex roots: $\alpha+i\beta$ and $\alpha-i\beta$.

We claim that $f_1(x)=e^{\alpha x}\cos(\beta x)$ and $f_2(x)=e^{\alpha x}\sin(\beta x)$ will be linearly independent solutions to the ODE.

Since $\alpha+i\beta$ and $\alpha-i\beta$ are roots we know the characteristic equation factors as follows:

$$r^2+br+c=(r-(\alpha+i\beta))(r-(\alpha-i\beta))$$
$$=r^2-2\alpha r+\alpha^2+\beta^2$$

Thus, $b=-2\alpha$ and $c=\alpha^2+\beta^2$.

Let's show $f_1(x)=e^{\alpha x}\cos(\beta x)$ solves the ODE.

We have

$$f_1(x)=e^{\alpha x}\cos(\beta x)$$

$$f_1'(x) = \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x)$$

$$\begin{aligned} f_1''(x) &= \alpha^2 e^{\alpha x}\cos(\beta x) - \alpha\beta e^{\alpha x}\sin(\beta x) \\ &\quad - \beta\alpha e^{\alpha x}\sin(\beta x) - \beta^2 e^{\alpha x}\cos(\beta x) \\ &= \alpha^2 e^{\alpha x}\cos(\beta x) - 2\alpha\beta e^{\alpha x}\sin(\beta x) \\ &\quad - \beta^2 e^{\alpha x}\cos(\beta x) \end{aligned}$$

Plugging these into the ODE gives

$$\begin{aligned} f_1'' + bf_1' + cf_1 &= f_1'' - 2\alpha f_1' + (\alpha^2 + \beta^2) f_1 \\ &= \alpha^2 e^{\alpha x}\cos(\beta x) - 2\alpha\beta e^{\alpha x}\sin(\beta x) - \beta^2 e^{\alpha x}\cos(\beta x) \\ &\quad - 2\alpha^2 e^{\alpha x}\cos(\beta x) + 2\alpha\beta e^{\alpha x}\sin(\beta x) \\ &\quad + \alpha^2 e^{\alpha x}\cos(\beta x) + \beta^2 e^{\alpha x}\cos(\beta x) \\ &= (\alpha^2 - \beta^2 - 2\alpha^2 + \alpha^2 + \beta^2)\cos(\beta x) \\ &\quad + (-2\alpha\beta + 2\alpha\beta)\sin(\beta x) \\ &= 0 \end{aligned}$$

So, $f_1(x) = e^{\alpha x}\cos(\beta x)$ solves the ODE.

Similarly you can check that $f_2(x) = e^{\alpha x}\sin(\beta x)$ solves the ODE. Let's make sure these solutions are linearly independent.

We have

$$W\left(e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\right)$$

$$= \begin{vmatrix} e^{\alpha x}\cos(\beta x) & e^{\alpha x}\sin(\beta x) \\ \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x) & \alpha e^{\alpha x}\sin(\beta x) + \beta e^{\alpha x}\cos(\beta x) \end{vmatrix}$$

$$= \alpha e^{2\alpha x}\cos(\beta x)\sin(\beta x) + \beta e^{2\alpha x}\cos^2(\beta x)$$
$$- \alpha e^{\alpha x}\sin(\beta x)\cos(\beta x) + \beta e^{2\alpha x}\sin^2(\beta x)$$

$$= \beta e^{2\alpha x}\left(\cos^2(\beta x) + \sin^2(\beta x)\right)$$

$$= \beta e^{2\alpha x} \neq 0$$ for any x since $\beta \neq 0$.

Conclusion: Every solution to the ODE is of the form

$$y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Ex: The ODE

$$y'' - 4y + 13y = 0$$

has charcteristic equation

$$r^2 - 14r + 13 = 0$$

which has two complex roots

$$r = \underbrace{2 + 3i}_{\alpha + i\beta},\ \underbrace{2 - 3i}_{\alpha - i\beta}$$

$$\alpha = 2,\ \beta = 3$$

we saw this earlier

Thus every solution to $y'' - 4y + 13y = 0$ is of the form

$$y_h = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$$

where c_1, c_2 are constants

Summary: Consider the second order linear, homogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = 0 \qquad (*)$$

where a_0, a_1, a_2 are real number constants and $a_2 \neq 0$.

Case 1: If the characteristic equation of $(*)$ has two distinct real roots r_1, r_2, then every solution of $(*)$ is of the form

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: If the characteristic equation of $(*)$ has one real repeated root r, then every solution to $(*)$ is of the form

$$y_h = c_1 e^{rx} + c_2 x e^{rx}$$

Case 3: If the characteristic equation of $(*)$ has two complex roots $\alpha + i\beta$ and $\alpha - i\beta$ then every solution to $(*)$ is of the form

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$